Roll No. ....

DD-459 (SE)

M. A./M. Sc. (Second Semester)
EXAMINATION, November, 2020

MATHEMATICS

Paper First

(Advanced Abstract Algebra—II)

Time : Three Hours

Maximum Marks : 80

Note : Attempt any *two* parts from each question. All questions carry equal marks.

Unit—I

1. (a) Let M be a (finitely generated unital) free R-module with a basis $\{e_1, e_2, ... e_n\}$, then show that $M \simeq R^n$.

   (b) Prove that R-module M is noetherian if and only if every submodule of M is finitely generated.

   (c) State and prove Wedderburn-Artin theorem.

Unit—II

2. (a) For $T \in$ Hom (U, V), $\alpha \in F$, then show that $\alpha T \in$ Hom (U, V).

   (b) Let A be an algebra with unit element, over F. Then show that A is isomorphic to a subalgebra of A(V) for some vector space V over F.

(c) Let U and V be vector spaces over a field F, then show that Hom_F (U, V) $\simeq F^{m \times n}$ as vector space over F.

Unit—III

3. (a) Show that if $W \subset V$ be invariant under $T \in A$ (V), then T induces a linear transformation T on the quotient space V/W, defined by $(V + W)\,\bar{T} = VT + W$. If T satisfies the polynomial $q(x) \in F[x]$, then $\bar{T}$ also satisfy q (x). If P_1 (x) is the minimal polynomial for $\bar{T}$ over F and if P (x) is that for T then $p_1(x) \mid p(x)$.

(b) Let the linear transformation $T \in AF(V)$ be nilpotent, then show that $\alpha_0 + \alpha_1 T + \cdots + \alpha_m T^m$, where $\alpha i \in F$, $0 \le i \le m$ is invertible if $\alpha_0 \ne 0$.

(c) Show that two nilpotent linear transformations S, $T \in A(V)$ are similar if and only if they have the same invariants.

Unit—IV

4. (a) Obtain the Smith normal form and rank for the following matrix over a PID R :

$$\begin{bmatrix} 0 & 2 & -1 \\ -3 & 8 & 3 \\ 2 & -4 & -1 \end{bmatrix}, R = Z$$

(b) Obtain the Smith normal form and rank for the following matrix over PID R :

$$\begin{bmatrix} -x-3 & 2 & 0 \\ 1 & -x & 1 \\ 1 & -3 & -x-2 \end{bmatrix}, \text{ where } R = Q\,[x]$$

(c) Find the abelian group generated by (x_1, x_2, x_3) subject to $5x_1 + 9x_2 + 5x_3 = 0$, $2x_1 + 4x_2 + 2x_3 = 0$, $x_1 + x_2 - 3x_3 = 0$.

Unit—V

5. (a) Reduce the following matrix A to a rational canonical form :

$$\begin{bmatrix} -3 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & -3 & -2 \end{bmatrix}$$

(b) Let V be a finite dimensional vector space over a field F, and let $T \in \operatorname{Hom} F(V, V)$. Suppose $f(x) = g(x)\, h(x)$ is a factorization of $f(x)$ in $f(x)$ such that gcd $(g(x), h(x)) = 1$, then show that $f(T) = \hat{0}$ if and only if $V = \ker g(T) \textcircled{f} \ker h(J)$.

(c) Find invariant factors, elementary divisors and the Jordan canonical form of the following matix :

$$\begin{bmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{bmatrix}$$

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DD-462 (SE)

M. A./M. Sc. (Second Semester) EXAMINATION, November, 2020

MATHEMATICS

Paper Fourth

(Advanced Complex Analysis—II)

Time : Three Hours

Maximum Marks : 80

Note : Attempt *two* parts from each Unit. All questions carry equal marks.

Unit—I

1. (a) If $|z| \leq 1$ and $p \geq 0$, then show that :

$$\left|1 - \mathrm{E}_p(z)\right| \leq |z|^{p+1}$$

(b) Let $\mathrm{S} = \{z \in \mathrm{C} : a \leq \mathrm{Re}\, z \leq \mathrm{A}\}$, where $0 < a < \mathrm{A} < \infty$. Then show that for every $\in > 0$ there is a number K such that for all z in S :

$$\left|\int_{\alpha}^{\beta} e^{-t}\, t^{z-1}\, dt\right| < \in$$

whenever $\beta > \alpha > k$.

(c) Let r be a rectifiable curve and let K be a compact set such that $\mathrm{K} \cap \{r\} = \phi$. Let f be a continuous function on $\{r\}$ and let $\in > 0$ be given. Then show that there is a rational function R(z) having all its poles on $\{r\}$ and such that :

$$\left| \int_r \frac{f(w)}{w - z} dw - \mathrm{R}(z) \right| < \in$$

for all z in K.

Unit—II

2. (a) Let $r : [0,1] \to \mathrm{C}$ be a path and let $\{(f_t, \mathrm{D}_t) : 0 \le t \le 1\}$ be an analytic continuation along r. For $0 \le t \le 1$ let R(t) be the radius for convergence of the power series expansion of f_t about $z = r(t)$. Then show that either $\mathrm{R}(t) \equiv \infty$ or **R** : [0, 1] $\to$ (0, ∞) is continuous.

(b) Show that the function $f_1(z) = 1 + 2 + 2^2 + 2^3 + ...$ can be obtained outside the circle of convergence of the power series.

(c) Define Analytic Continuation. If the radius of convergence of the power series :

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is non-zero finite, then show that f (z) has at least one singularity on the circle of convergence.

Unit—III

3. (a) Define Poisson kernel. Show that the Poisson kernel $\mathrm{P}_r(\theta)$ satisfies the following properties :

(i) $$\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{P}_r(\theta)\, d\theta = 1$$

(ii) $\mathrm{P}_r(\theta) > 0$ for all θ, $\mathrm{P}_r(-\theta) = \mathrm{P}_r(\theta)$ and P_r is periodic in θ with period 2π.

(b) Let G be a region and $f: \partial_\infty G \to \mathbf{R}$ a continuous function. Then show that $u(z)$ = Sup $\{\phi(z) : \phi \in P(f, G)\}$ defines a harmonic function u in G.

(c) To state and prove Harnack's theorem for harmonic functions.

Unit—IV

4. (a) If $f(z)$ is analytic within and on the circle r such that $|z| = R$ and if $f(z)$ has zeros at the points $a_i \neq 0$, $(i = 1, 2, ..., m)$ and poles at $b_j \neq 0$, $(j = 1, 2, ..., n)$ inside r, multiple zeros and poles-being repeated, then show that :

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(\mathrm{Re}^{i\theta}\right| d\theta = \log|f(\mathrm{o})| + \sum_{i=1}^{m} \log \frac{R}{|a_i|} - \sum_{j=1}^{n} \log \frac{R}{|b_j|}$$

(b) Define order of an Entire Function. Find the order of polynomial $P(z) = a_0 + a_1 z + ... + a_n z^n, a_n \neq 0$.

(c) Use Hadamard's factorization theorem to show that :

$$\sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

Unit—V

5. (a) Define Bloch's constant. Let f be an analytic function in a region containing the closure of the disc $D = \{z : |z| < 1\}$ and $f(0) = 0$, $f'(0) = 1$. Then show that $f(D)$ contains a disc of radius L.

(b) For each α and β, $0 < \alpha < \infty$ and $0 \le \beta \le 1$, there is a constant $C(\alpha, \beta)$ such that if f is an analytic function defined in some simply connected region containing $\overline{B}(0,1)$ that omits the values 0 and 1 and such that $|f(0)| \le \alpha$; then show that :

$$|f(z)| < C(\alpha, \beta) \text{ for } |z| \le \beta.$$

(c) Let $f \in y$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then show that :

(i) $|a_2| \le 2$

(ii) $f(U) \supset D\left(0; \frac{1}{4}\right)$

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DD-463 (SE)

M. A./M. Sc. (Second Semester) EXAMINATION, November, 2020

MATHEMATICS

Paper Fifth

(Advanced Discrete Mathematics—II)

Time : Three Hours

Maximum Marks : 80

Note : Attempt all the *five* questions selecting *two* parts from each Unit. All questions carry equal marks.

Unit—I

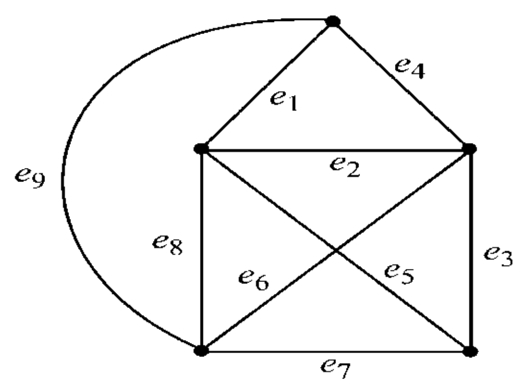
1. (a) Show that the number of vertices of odd degree in a graph is always even.

   (b) Prove that a connected graph G is a Euler graph if and only if it can be decomposed into circuits.

   (c) Show that every tree has either one or two centres.

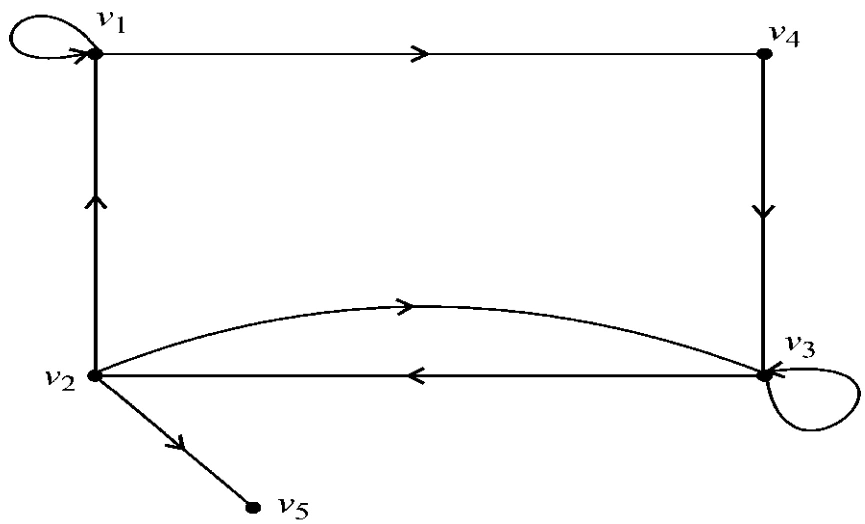
Unit—II

2. (a) Draw all spanning trees of the following graph :



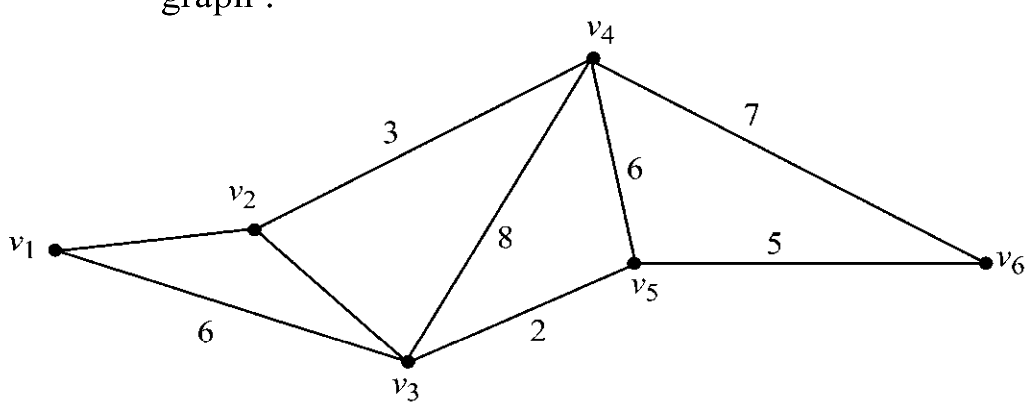
   (b) Define cut-sets. Prove that every cut-set in a connected graph G contains at least one branch of every spanning tree of G.

(c) Define adjacency matrix with respect to matrix representation of graph and find the adjacency matrix of the following digraph :

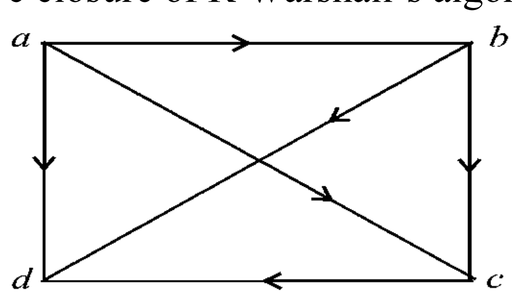


Unit—III

3. (a) Find the shortest path from v_1 to v_6 in the following graph :



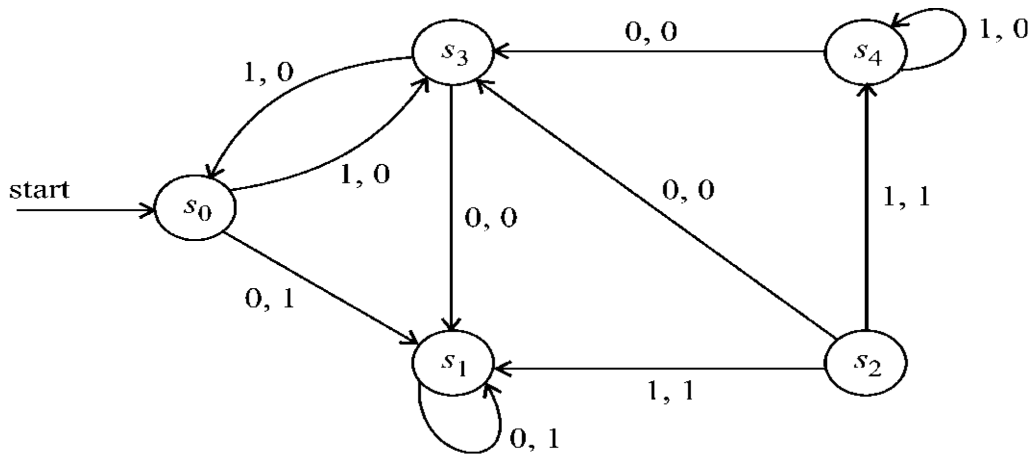
(b) Let R = {(*a*, *b*), (*a*, *c*), (*a*, *d*), (*b*, *c*), (*b*, *d*), (*c*, *d*)} be a relation on the set A = {*a*, *b*, *c*, *d*}. Find the transitive closure of R Warshall's algorithm.



(c) Explain traversal of a tree algorithm (any *two*).

Unit—IV

4. (a) Construct the state table for the finite state machine with the diagram shown below :



(b) Construct a finite state machine that accepts the language :

L = {all binary sequences that end with digit 101}

L = {011, 1011, 0011, 10011, 11011, ...}

(c) Reduce (or minimize) the machine whose state table is given below :

State	Input		Output
	0	1	
⇒ S_0	S_3	S_6	1
S_1	S_4	S_2	0
S_2	S_4	S_1	0
S_3	S_2	S_0	1
S_4	S_5	S_0	1
S_5	S_3	S_5	0
S_6	S_4	S_2	1

Unit—V

5. (a) Design a turing machine to recognize all strings consisting of even number of *a*'s.

   (b) State the difference between Moore machine and Mealy machine.

   (c) Construct a non-deterministic finite automata which accepts 1100 only.

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DD-461 (SE)

M. A./M. Sc. (Second Semester) EXAMINATION, November, 2020

MATHEMATICS

Paper Third

(General and Algebraic Topology)

Time : Three Hours

Maximum Marks : 80

Note : Attempt any *two* parts from each Unit. All questions carry equal marks.

Unit—I

1. (a) Let (X, T) be the product topological space of an indexed family of topological spaces $(X_i, T_i) : i \in I$ and let Y be any topological space. Then prove that a function $f : Y \rightarrow X$ is continuous (w. r. t. the product topology on X) if and only if for each $i \in I$, the composition $\pi_i \circ f : Y \rightarrow X_i$ is continuous. 8

   (b) Define completely regular space. Prove that a product of topological spaces is completely regular if and only if each co-ordinate space is so. 8

(c) Define T_1-space with an example. Prove that the product space $X = \Pi \ X_i : i \in I$ is a T_1-space if and only if each co-ordinate space is a T_1-space. 8

Unit—II

2. (a) Write short notes on the following : 8

   (i) Connected space

   (ii) Metrisable space

   (iii) Compact space

   (iv) Second countable space

(b) Define path connected space with an example. Show that a product of topological spaces is path connected if and only if each co-ordinate space is path connected. 8

(c) Define first countable space. Prove that the product space $X = \Pi \ X_i : i \in I$ is first countable at x if and only if for each $i \in I$, X_i is first countable at $\pi_i \ x$ and all except countably many i's, X_i is the only neighbourhood of $\pi_i \ x$ in X_i. 8

Unit—III

3. (a) Let $f_i : X \to Y_i : i \in I$ be a family of functions which distinguishes points from closed sets in X. Then show that the corresponding evaluation function $e : X \to \prod_{i \in I} Y_i$ is open when regarded as a function from X onto $e \ X$. 8

(b) Define paracompact space with an example. Show that every paracompact space is normal. 8

(c) State and prove Urysohn Metrization theorem. 8

Unit—IV

4. (a) (i) Define the following : 4

(I) Convergence of a net

(II) Filter

(III) Cluster point of a filter

(IV) Ultrafilter

(ii) If a topological space X is compact, then prove that every net in X has a cluster point in X. 4

(b) (i) Let X be any infinite set and $F = \{A \subseteq X : X - A \text{ is finite}\}$, then prove that F is a filter on X. 4

(ii) Show that a filter F on a set X is an ultrafilter if and only if for any $A \subset X$ either $A \in F$ or $X - A \in F$. 4

(c) Prove that every filter is contained in an ultrafilter. 8

Unit—V

5. (a) (i) If $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic. Then show that $k \circ h$ and $k' \circ h'$ are homotopic. 4

(ii) Show that in a fundamental group the map "α-hat", that is $\hat{\alpha}$ is a group isomorphism. 4

(b) Define a covering map. Show that the map $p : R \to S^1$ given by the equation : 8

$$p\ x = \cos 2\pi x, \sin 2\pi x$$

is a covering map.

(c) State and prove Fundamental theorem of algebra. 8

Roll No. ....

DD-460 (SE)

M. A./M. Sc. (Second Semester)
EXAMINATION, November, 2020

MATHEMATICS

Paper Second

(Real Analysis—II)

Time : Three Hours

Maximum Marks : 80

Note : Solve any *two* parts from each question. All questions carry equal marks.

Unit—I

1. (a) Suppose $C_n \geq 0$ for $n = 1, 2, 3, \ldots\ldots$, $\Sigma\ C_n$ converges S_n is a sequence of distinct points in (a, b) and $\alpha\ \ a\ = \sum_{n=1}^{\infty} C_n\, I\ \ x - S_n$. Let f be continuous on a, b . Then prove that :

$$\int_a^b f\, d\alpha = \sum_{n=1}^{\infty} C_n f\ \ S_n$$

(b) Find the value of :

$$\int_1^2 x\, d\ \ \ x$$

(c) Let $f \in R$ on $[a,b]$. For $a \le x \le b$, put $F(x) = \int_a^x f(t)\,dt$. Then prove that F is continuous on $[a,b]$, if f is continuous at a point x_0 of $[a,b]$, then prove that F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Unit—II

2. (a) The union of a finite number of measurable sets is measurable.

(b) State and prove the Fatou's Lemma.

(c) State and prove that Lebesgue Dominated Convergence theorem.

Unit—III

3. (a) Let (X, S, μ) be a σ-finite measure space, Σ a semiring of sets such that $S \subset \Sigma \subset \beta$ and $\bar{\mu}$ a measure on Σ. If $\bar{\mu} = \mu$ on S, then prove that $\bar{\mu} = \mu^*$ on Σ. In particular, μ^* is the only extension of μ to a measure on β.

(b) Prove that the set function μ^* is an outer measure.

(c) Let the sets $E_1, E_2, \ldots\ldots\ldots\ldots, E_n$ be disjoint and measurable. Then prove that :

$$\mu^*\left[A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right] = \sum_{i=1}^{n} \mu^*(A_n E_i)$$

holds, for every subset A of X.

Unit—IV

4. (a) Let $f : \mathrm{R} \to \mathrm{R}$ be defined by :

$$f\ x = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then find $D^+ f\ 0 = D^- f\ 0 = 1$ and $D_+ f\ 0 = D_- f\ 0 = -1$. Consequently, f is not differentiable at $x = 0$.

(b) State and prove the Fundamental theorem of Integral Calculus.

(c) (i) Define Four derivatives.

(ii) If the function f assumes its maximum at C, show that $D^+ f\ C \leq 0$ and $D_- f\ C \geq 0$.

Unit—V

5. (a) State and prove the Jordan Decomposition theorem.

(b) State and prove the Minkowski's Inequality.

(c) State and prove Riesz-Fischer theorem.